# Shape Factors for Conductive Heat Flow

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**Application of shape factors to problems of conductive heat flow eliminates the need for lengthy calculations by numerical approximation methods. Shape factors for several systems, determined by electrical analogues, are given in the accompanying article.**

In differential form the Fourier equation for conductive heat flow is

$$dq = -k(dt/dx)\,dA \qquad (1)$$

When the conductivity $k$ is constant, Equation (1) may be integrated for steady flow between one boundary at temperature $t_1$ and another at $t_2$ to give

$$q = -k(\bar{A}/\bar{x})(t_2 - t_1)$$
$$= -kF_L(t_2 - t_1) \qquad (2)$$

Here $F_L$ is the Langmuir *shape factor* (*4*, *5*), the ratio of the average cross-sectional area for heat flow to the average effective distance for travel. Equation (2) is valid for any system in which $k$ is constant, each boundary is at a uniform temperature, and there is no production or absorption of heat between the boundaries of the system. The rate of heat flow may then be calculated from the conductivity and total temperature drop, provided the shape factor $F_L$ is known.

Rigorous calculation of $F_L$ is easy for a few simple shapes, possible but difficult for certain others, and impossible for a great many. Even a simple problem such as two-dimensional flow between concentric squares defies rigorous mathematical analysis. Several methods have been devised for estimating shape factors for such systems, including numerical approximation methods, graphical methods, and electrical and fluid-flow devices (*3*, *6*).

Unfortunately the methods are often tedious and inconvenient to use in practical problems. Once the shape factor for a given system has been accurately established, however, and presented in an equation or chart, it can be used directly to compute rates of transfer. The tedious calculations or measurements are no longer needed. Langmuir, *et al.* (*5*), presented empirical equations for two- and three-dimensional flow through bodies bounded by rectangular prisms or right-rectangular parallelepipeds, with corresponding inner and outer surfaces parallel and, in any given system, the same distance apart. Equations for several other shapes are given by McAdams (*6*). Andrews (*1*) derived equations for a number of shapes and checked them experimentally.

This study was undertaken to establish shape factors for several systems of industrial importance which have received little or no attention in previous work.

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### MEASUREMENT OF SHAPE FACTORS

Shape factors were measured by the method described by Andrews (*1*) for two-dimensional figures. This involves measurement of the electrical resistance of the figure and comparison with the resistance of a standard reference figure of known shape factor. The conducting medium was a sheet of Teledeltos Recording Paper Type L. The figures were carefully outlined with General Cement Silver Print, a silver paint made for use in printed circuits. Electrical connections were made with spade lugs cemented to the boundaries of the figures with an excess of silver paint. The reference figure consisted of two concentric circles.

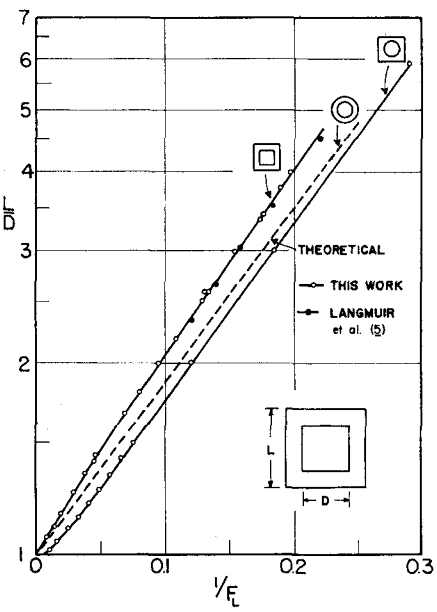


**Fig. 1. Shape factors for concentric squares, concentric circles, and a circle within a square.**

The conductivity of the paper was not entirely constant but varied with direction and from region to region. The resistivity, on the average, was 11% greater in the direction of the width of the roll of paper than in the direction of its length. The standard deviation of the resistivities from region to region in the longitudinal direction was 1.7%; in the transverse direction it was 2.0%. In addition, the resistivity varied with the humidity of the surrounding air: it rose by about 10% when the relative humidity changed from 20 to 80%. The effect of these variations on the results was very small because in most of the tests both the unknown and the reference figures were symmetrical about several axes and comparison with a standard figure eliminated the effect of changes in humidity.

The figures tested were concentric squares, a circle inside a square and concentric with it, and a circle inside and concentric with a rectangle. The first might be applied to heat flow through the walls of a tall square chimney of uniform wall thickness; the second two might represent a cylinder or pipe enclosed in a square or rectangular duct, filled with a conducting or insulating medium. In the studies of a circle within a rectangle, the figure was drawn with its longest sides at an angle of 45 deg. with the longitudinal direction of the roll of conducting paper. With each figure sets of runs were made, with the relative sizes of the inner and outer boundaries varied: in one set the outer boundary was kept constant and the inner boundary varied; in a second set the reverse was true. In most of the last set of runs the long side of the rectangle was 1.2 times as long as the short side, and the diameter of the circle was varied. In a few runs the diameter of the circle was held at half the length of the short side, and the length of the other side of the rectangle was varied.

### RESULTS

The results for concentric squares and for a circle within a square are shown in Figure 1. The ratio of the length of one side of the outer square $L$ to the side or diameter of the inner boundary $D$ is plotted against the reciprocal of the shape factor $1/F_L$. The theoretical graph for concentric circles, where $L$ is now the diameter of the outer circle, is included for comparison. Shape factors for a circle within a rectangle are given in Figures 2 and 3. These shape factors are all for two-dimensional flow in a system of unit length in the third dimension.

#### Concentric Squares

The graph for concentric squares (Figure 1) consists of two linear branches joined by a short section of gentle curvature. At values of $L/D$ greater than 1.7 the line is straight and nearly parallel to that for concentric circles. In this region it is defined by the empirical equation

$$F_L = \frac{a}{\log \dfrac{L}{D} - b} \qquad (3)$$

where

$$a = 2.92 \pm 0.05$$

$$b = 0.0233 \pm 0.0007$$

The values of $a$ and $b$ were found by the method of least squares, the uncertainties being computed at the 90% confidence limits. The corresponding correlation coefficient is 0.999950.

The theoretical equation for concentric circles is

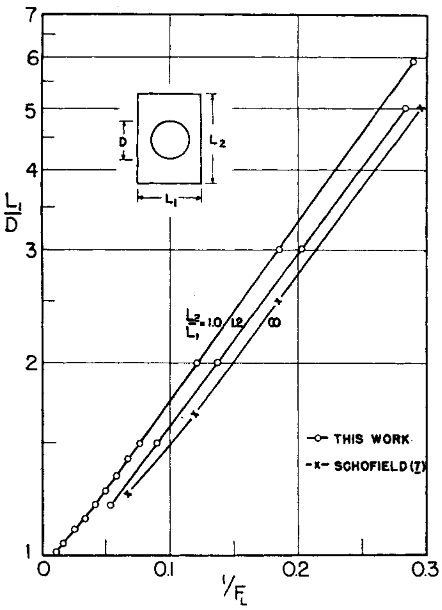


**Fig. 2. Shape factors for a circle inside and concentric with a rectangle.**

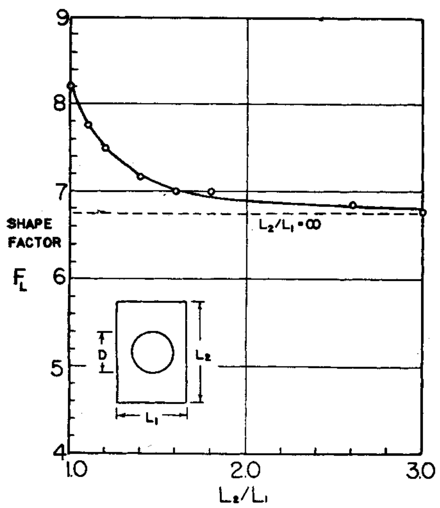


**Fig. 3. Variation of shape factor with $L_2/L_1$ for a circle within a rectangle, ($L_1/D = 2$).**

$$F_L = 2\pi/\ln (L/D)$$
$$= 2.728/\log (L/D) \qquad (4)$$

When $L/D$ for concentric squares is less than 1.35 the distance between the squares is small, and the flow approximates that between plates of equal area. The measured shape factors for this region agree closely with those calculated on the basis of the logarithmic mean of the perimeters of the inner and outer boundaries of the figure and are given by the equation

$$F_L = 3.47/\log (L/D) \qquad (5)$$

When $L/D$ is between 1.35 and 1.7, neither Equation (3) nor (5) exactly describes the graph, but the maximum error in using either one when $L/D$ is 1.4 is about 6%. Equation (3) should be used when $L/D$ is greater than 1.4 and Equation (5) when it is smaller than 1.4.

Data given by Langmuir, *et al.* (*5*), for concentric squares are shown in Figure 1 as solid circles. The points are slightly below the graph defined by Equation (3). They were obtained by measuring the electrical conductivity of a copper sulfate solution and are believed to be somewhat less accurate than those reported here. To check the accuracy of the present work, the shape factors for two values of $L/D$ were calculated by the relaxation method (*2*), with an IBM 650 computer. The results are given in Table 1. The present measurements are thus believed to be accurate to within 0.5%.

TABLE 1. COMPARISON OF COMPUTED AND MEASURED SHAPE FACTORS

| $L/D$ | Number of points in network | $F_L$ computed by relaxation | $F_L$ from Eq. (3) | Difference, % |
|---|---|---|---|---|
| 2.0 | 144 | 10.59 | 10.51 | 1.0 |
| 2.0 | 240 | 10.49 | 10.51 | 0.1 |
| 3.5 | 216 | 5.58 | 5.61 | 0.5 |

### Circle Within a Square

The graph for a circle within a square, also in Figure 1, is almost straight and parallel with that for concentric circles when $L/D$ exceeds 1.2. It is below the graph for concentric circles, instead of above it as is the graph for concentric squares. The linear portion is described by the empirical equation

$$F_L = \frac{c}{\log \dfrac{L}{D} + d} \qquad (6)$$

where

$$c = 2.79 \pm 0.010$$

$$d = 0.0360 \pm 0.0007$$

The uncertainties in $c$ and $d$ were also estimated at the 90% confidence limits. The correlation coefficient is 0.999985.

At low values of $L/D$ the graph bends toward the origin, slowly at first, and then very sharply. The shape factor becomes infinite when $L/D$ reaches 1, that is, when the inner circle touches the sides of the square.

### Circle Within a Rectangle

Figures 2 and 3 show the results for a circle inside a rectangle, with the center of the circle at the intersection of the diagonals. Figure 2 applies to systems in which the long sides of the rectangle are 1.2 times the short sides ($L_2/L_1 = 1.2$). Also shown are the graphs for a circle within a square ($L_2/L_1 = 1.0$) and for a circle midway between infinite parallel plates ($L_2/L_1 = \infty$). This last graph is based on the calculations of Schofield (*7*) and represents the midpoints of the ranges he estimated for the shape factors. The graph for $L_2/L_1 = 1.2$ is straight and almost parallel with that for the circle within a square but is surprisingly close to that for a circle between infinite plates. This similarity is borne out by Figure 3, which shows how fast the shape factor diminishes as $L_2/L_1$ increases. Here $D$, the diameter of the circle, is always half the short side of the rectangle $L_1$. Once the long side of the rectangle is twice the short side, the shape factor is only about 2% greater than the asymptotic value for infinite parallel plates.

## NUMERICAL EXAMPLE

The example used by Emmons (*2*) to illustrate the relaxation method will be solved by the equations presented above. The problem is to compute the heat loss per foot of height through the walls of a square chimney when the inner opening is 24 in. on a side, and the walls are 10 in. thick. The inside surface is at a uniform temperature of 500°F.; the outside surface is at 100°F. The conductivity of the walls is $k$ B.t.u./(hr.)(sq. ft.)/(°F./ft.).

*Solution.*

$$D = 24 \text{ in.} \qquad L = 24 + 20 = 44 \text{ in.}$$
$$L/D = 1.833$$

From Equation (3),

$$F_L = 2.92/(\log 1.833 - 0.0233) = 12.17$$

Thus

$$q = 12.17k(500 - 100) = 4868k \text{ B.t.u./hr.}$$

The result Emmons obtained by the relaxation method was 4,848$k$ B.t.u./hr.

## NOTATION

- $a$ = empirical constant in Equation (3)
- $A$ = area normal to direction of flow, sq. ft.; $\bar{A}$ = effective mean area
- $b$ = empirical constant in Equation (3)
- $c, d$ = empirical constants in Equation (6)
- $D$ = diameter or side of inner boundary, ft.
- $F_L$ = Langmuir shape factor, $\bar{A}/\bar{x}$
- $k$ = thermal conductivity, B.t.u./(hr.)(sq. ft.)/(°F./ft.)
- $L$ = length of side or diameter of outer boundary, ft.; $L_1$ = length of short side of rectangle; $L_2$ = length of long side of rectangle
- $q$ = rate of heat flow, B.t.u./hr.
- $t$ = temperature, °F.; $t_1$ of hot boundary; $t_2$ of cold boundary
- $x$ = distance in direction of transfer, ft.; $\bar{x}$ = mean effective distance for transfer

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